Phase Fluctuations and Single Fermion Spectral Density in 2D Systems with Attraction

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Abstract

The effect of static fluctuations in the phase of the order parameter on the normal and superconducting properties of a 2D system with attractive four-fermion interaction is studied. Analytic expressions for the fermion Green's function, its spectral density, and the density of states are derived in the approximation where the coupling between the spin and charge degrees of freedom is neglected. The resulting single-particle Green's function clearly demonstrates a non-Fermi liquid behavior. The results show that as the temperature increases through the 2D critical temperature, the width of the quasiparticle peaks broadens significantly.

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I. INTRODUCTION

One of the most convincing manifestations of the difference between the BCS scenario and superconductivity in the cuprates is the pseudogap, or the depletion of a single particle spectral weight around the Fermi level [1]. This is observed mainly in the underdoped cuprates where the pseudogap opens in the normal state as the temperature T decreases below the crossover temperature T^* and extends over a wide range of T.

Due to the complex nature of cuprate systems, there are a number of theoretical explanations for the pseudogap behavior. One of them is based on the model of a nearly antiferromagnetic Fermi liquid [2]. Another possible explanation relates the pseudogap to spin- and/or charge-density waves [3]. A third direction, which we take in this paper, argues that precursor superconducting fluctuations may be responsible for the pseudogap phenomena. Indeed an incoherent pair tunneling experiment [4] proposed recently may allow one to answer whether the superconducting fluctuations are really responsible for the pseudogap behavior. Furthermore, one cannot exclude the possibility that the pseudogap is the result of a combination of various mechanisms, e.g., both spin and superconducting fluctuations.

The precursor superconducting fluctuations have recently been extensively studied using different approaches. In most cases, the attractive 2D or 3D Hubbard model was considered. In particular this model has been studied, both analytically [5–7] and numerically [8–11], in the conserving T-matrix approximation that is "Φ derivable" in the sense of Baym [12]. The non "Φ derivable" T-matrix approximation was considered in [13]. In this approach, the pseudogap is related to the resonant pair scattering of correlated electrons above T_c. For the d-wave pairing, the pseudogap was also studied in [14] (for a review, see [15]) and Monte Carlo simulations for the 2D attractive Hubbard model were performed in [16].

It is known, however, that while the T-matrix approximation provides an adequate description of 3D systems at all temperatures, including the superconducting state with a long-range order, it fails (see, for example, [9]) to describe the Berezinskii-Kosterlitz-Thouless (BKT) transition into the state with an algebraic order, which is only possible in 2D systems. This is why, in most of the papers cited above, the T-matrix approximation was used to study either 3D systems [5,6,10,13] or 2D systems above T_c [8,9,14,15] in order to avoid the BKT transition, even though it is generally accepted that 2D models are more relevant for the description of cuprates [17].

Of course, the superconducting transition itself is not of the BKT type, because even a weak interplanar coupling produces a transition in the $d = 3$ XY universality class, sufficiently close to the transition temperature. Outside the transition region, however, the low-energy physics is governed by vortex fluctuations [18], and one can expect the 2D model to be especially relevant for the description of the pseudogap phase. This was confirmed for the quasi-2D model [19] (see also [20]).

Regarding the pseudogap, it is sufficient to consider the case where $T > T_c$. However, one definitely needs an approach different from the T-matrix if one wants to study the 2D theory for the entire temperature range and wants to connect the pseudogap to the superconducting gap. An alternative approach overcoming the above difficulty was proposed in [21–23]. For a 2D system, one should rewrite the complex order field $\Phi(x)$ in terms of its modulus $\rho(x)$ and its phase $\theta(x)$ as $\Phi(x) = \rho(x)\exp[i\theta(x)]$, which was originally suggested by Witten in the context of 2D quantum field theory [24]. It is impossible to obtain $\Phi \equiv \langle\Phi(x)\rangle \neq 0$ at finite

T because this would correspond to the formation of symmetry breaking homogeneous long-range order, which is forbidden by the Coleman—Mermin—Wagner—Hohenberg (CMWH) theorem [25]. However, it is possible to obtain $\rho \equiv \langle\rho(x)\rangle \neq 0$ with $\Phi = \rho\langle\exp[i\theta(x)]\rangle = 0$ at the same time because of random fluctuations of the phase $\theta(x)$ (i.e., because of transverse fluctuations of the order field originating in the modulus conservation principle [26]). We stress that $\rho \neq 0$ does not imply any long-range superconducting order (which is destroyed by phase fluctuations) and, therefore, does not contradict the abovementioned theorem.

For the simple model studied in [21,22], there are three regions in the 2D phase diagram. The first one is the superconducting (here, BKT) phase with $\rho \neq 0$ at $T < T_{\mathrm{BKT}}$, where T_{BKT} is the BKT transition temperature, which plays the role of T_c in pure 2D superconducting systems. In this region, there is an algebraic order or a power law decay of the $\langle\Phi^*\Phi\rangle$ correlations. The second region corresponds to the so-called pseudogap phase ($T_{\mathrm{BKT}} < T < T_\rho$), where T_ρ is the temperature at which ρ is supposed to become zero. In this phase, ρ is still non-zero, but the above correlations decay exponentially. The third is the normal (Fermi-liquid) phase at $T > T_\rho$, where $\rho = 0$. Note that Φ and all the symmetry violating correlators like $\langle\Phi(x)\Phi(0)\rangle$ vanish everywhere.

The proposed description of the phase fluctuations and the BKT transition is very similar to that given by Emery and Kivelson [27]. However, the field $\rho(x)$ does not appear the phenomenological approach of [27], while in the present microscopic approach it occurs naturally. We also mention here the application of similar ideas to the 3D case [28], where instead of the 2D temperature T_{BKT} one has the phase transition temperature in the 3D XY-model, T_c^{XY}.

The main quantity of interest in the present paper is the one-fermion Green's function and the associated spectral function $A(\omega, \mathbf{k}) = -(1/\pi)\mathrm{Im}G(\omega + i0, \mathbf{k})$. The second quantity, being proportional to the intensity of the angle-resolved photoemission spectrum (ARPES) [29], encodes information about the pseudogap and quasiparticles. Following the approach of Refs. [21–23] the Green's function for the charged (physical) fermions is given by the convolution (in momentum space) of the propagator for neutral fermions (which has a gap $\rho \neq 0$) and the Fourier transform of the phase correlator $\langle\exp(i\tau_3\theta(x)/2)\exp(-i\tau_3\theta(0)/2)\rangle$.

Thus, the approximation employed here assumes the absence of coupling between spin and charge degrees of freedom; this can be taken into account at the next stage of approximation. We demonstrate that the quasiparticle spectral function broadens considerably when passing from the superconducting to the normal state, as observed experimentally [29]. More importantly, the phase fluctuations result in a non-Fermi liquid behavior of the system both below and above T_{BKT}.

We note that the effect of classical phase fluctuations of the order field on the spectral properties of underdoped cuprates has also been analyzed by Franz and Millis [30]. Being experimentally motivated, they could show that the corresponding photoemission and tunneling data are well accounted for by a simple model where d-wave charge excitations are coupled to supercurrent fluctuations.

A brief overview of the paper is as follows: In Sec. II, we present the modulus-phase formalism for the fermion Green's function and explain why it is so important to use this formalism for the description of 2D models. In Sec. III, we obtain and discuss the Green's function of phase fluctuations both below and above T_{BKT}. This expression is then used in Sec. IV to derive the temperature and retarded fermion Green's functions. We show that

this Green's function exhibits a non-Fermi liquid behavior. In Sec. V, we obtain an analytic expression for the spectral density of the fermion Green's function and discuss this result in detail. The density of states (DOS) is considered in Sec. VI. Appendix A contains technical details on the calculation of the long-distance asymptotic behavior of the phase correlator. Appendix B contains the derivation of an alternative representation for the fermion Green's function which is useful in calculating the spectral density. The integrals for the DOS are given in Appendix C.

II. THE MODULUS—PHASE REPRESENTATION FOR THE FERMION GREEN'S FUNCTION

Our starting point is a continuum version of the two-dimensional attractive Hubbard model defined by the Hamiltonian density [21–23]

$$\mathcal{H} = \psi_\sigma^\dagger(x)\left(-\frac{\nabla^2}{2m} - \mu\right)\psi_\sigma(x) - V\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x), \tag{2.1}$$

where $x = \mathbf{r}, \tau$ denotes the space and imaginary time variables, $\psi_\sigma(x)$ is a fermion field with the spin $\sigma = \uparrow, \downarrow$, m is the effective fermion mass, μ is the chemical potential, and V is an effective local attraction constant; we take $\hbar = k_B = 1$. The model with the Hamiltonian density (2.1) is equivalent to the model with an auxiliary BCS-like pairing field, which can be written as

$$\mathcal{H} = \Psi^\dagger(x)\left[\tau_3\left(-\frac{\nabla^2}{2m} - \mu\right) - \tau_+\Phi(x) - \tau_-\Phi^*(x)\right]\Psi(x) + \frac{|\Phi(x)|^2}{V} \tag{2.2}$$

in terms of Nambu variables

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix}, \qquad \Psi^\dagger(x) = \begin{pmatrix} \psi_\uparrow^\dagger(x) & \psi_\downarrow(x) \end{pmatrix}, \tag{2.3}$$

where $\tau_\pm = (\tau_1 \pm i\tau_2)/2$, τ_3 are the Pauli matrices and $\Phi(x) = V\Psi^\dagger(x)\tau_-\Psi(x) = V\psi_\downarrow\psi_\uparrow$ is the complex order field.

We consider the full fermion Green's function in the Matsubara finite temperature formalism

$$G(x) = \langle \Psi(x)\Psi^\dagger(0)\rangle . \tag{2.4}$$

For the 3D case of the BCS theory, the frequency-momentum representation for (2.4) in the mean field approximation is known to be [31]

$$G(i\omega_n, \mathbf{k}) = -\frac{i\omega_n\hat{I} + \tau_3\xi(\mathbf{k}) - \tau_+\Phi - \tau_-\Phi^*}{\omega_n^2 + \xi^2(\mathbf{k}) + |\Phi|^2}, \tag{2.5}$$

where $\omega_n = (2n+1)\pi T$ is the odd (fermion) Matsubara frequency, $\xi(\mathbf{k})$ is the dispersion law of electrons evaluated from the chemical potential μ, and $\Phi \equiv \langle\Phi(x)\rangle$ is the complex order parameter.

A problem arises when one tries to apply Eq. (2.5) directly to 2D systems, since it has been proved (see [25]) that nonzero Φ values are forbidden. Nevertheless, one can assume that the modulus of the order parameter $\rho = |\Phi|$ has a nonzero value, while its phase $\theta(x)$ defined by

$$\Phi(x) = \rho(x) \exp\left[i\theta(x)\right] \tag{2.6}$$

is a random quantity. To be consistent with (2.6) one should also introduce the spin-charge variables for the Nambu spinors

$$\Psi(x) = \exp[i\tau_3\theta(x)/2]\Upsilon(x), \qquad \Psi^\dagger(x) = \Upsilon^\dagger(x)\exp[-i\tau_3\theta(x)/2], \tag{2.7}$$

where Υ is the neutral fermion field operator. The strategy of treating charge and spin (neutral) degrees of freedom as independent seems to be quite useful, and at the same time a very general feature of 2D systems [24,32].

Applying (2.7), we thus split the Green's function (2.4) into spin and charge parts

$$G_{\alpha\beta}(x) = \sum_{\alpha',\beta'} \mathcal{G}_{\alpha'\beta'}(x)\langle (e^{i\tau_3\theta(x)/2})_{\alpha\alpha'}(e^{-i\tau_3\theta(0)/2})_{\beta'\beta}\rangle, \tag{2.8}$$

where

$$\mathcal{G}_{\alpha\beta}(x) = \langle \Upsilon_\alpha(x)\Upsilon^\dagger_\beta(0)\rangle \tag{2.9}$$

is the Green's function for neutral fermions. Introducing the projectors $P_\pm = \frac{1}{2}(\hat{I} \pm \tau_3)$ we obtain

$$e^{i\tau_3\theta/2} = P_+ e^{i\theta/2} + P_- e^{-i\theta/2}, \quad e^{-i\tau_3\theta/2} = P_- e^{i\theta/2} + P_+ e^{-i\theta/2}, \tag{2.10}$$

so that (2.8) can be rewritten as

$$G(x) = \sum_{\alpha,\beta=\pm} P_\alpha \mathcal{G}(x) P_\beta \langle \exp[i\alpha\theta(x)/2]\exp[-i\beta\theta(0)/2]\rangle, \tag{2.11}$$

where $\alpha = \beta$ and $\alpha = -\beta$ correspond to the diagonal and non-diagonal parts of the Green's function, respectively.

For the frequency-momentum representation of (2.11) we have

$$G(i\omega_n, \mathbf{k}) = T\sum_{m=-\infty}^{\infty}\int \frac{d^2p}{(2\pi)^2}\sum_{\alpha,\beta=\pm} P_\alpha \mathcal{G}(i\omega_m, \mathbf{p}) P_\beta D_{\alpha\beta}(i\omega_n - i\omega_m, \mathbf{k}-\mathbf{p}), \tag{2.12}$$

where

$$\mathcal{G}(i\omega_m, \mathbf{k}) = \int_0^{1/T} d\tau \int d^2r \exp[i\omega_m\tau - i\mathbf{k}\mathbf{r}]\mathcal{G}(\tau, \mathbf{r}) \tag{2.13}$$

and

$$D_{\alpha\beta}(i\Omega_n, \mathbf{q}) = \int_0^{1/T} d\tau \int d^2r \exp\left(i\Omega_n\tau - i\mathbf{q}\mathbf{r}\right)\langle \exp[i\alpha\theta(\tau,\mathbf{r})/2]\exp[-i\beta\theta(0)/2]\rangle \tag{2.14}$$

is the correlator of phase fluctuations with even (boson) frequencies $\Omega_n = 2\pi n T$.

There is a good reason to believe (see [22]) that for T close to $T_{\rm BKT}$, the fluctuations of the order parameter modulus ρ (the so–called longitudinal fluctuations, which in fact correspond to carrier density fluctuations and undoubtedly must be taken into account in the very underdoped region) [33] are irrelevant and one can safely use the Green's function (2.13) of the neutral fermions in the mean-field approximation (compare with (2.5))

$$\mathcal{G}(i\omega_n, \mathbf{k}) = -\frac{i\omega_n \hat{I} + \tau_3 \xi(\mathbf{k}) - \tau_1 \rho}{\omega_n^2 + \xi^2(\mathbf{k}) + \rho^2}. \tag{2.15}$$

Here $\xi(\mathbf{k}) = \mathbf{k}^2/2m - \mu$ with $\mathbf{k}$ being a 2D vector and $\rho \equiv \langle \rho(x) \rangle$. Note that in [21,22], $\rho(x)$ was treated only in the mean-field approximation, which means that fluctuations in both $\rho(x)$ and $\theta(x)$ were neglected, and therefore a second-order phase transition was obtained at T_ρ. However, as stressed in Introduction, experimentally the formation of the pseudogap phase does not display any sharp transition and the temperature T^* observed in various experiments is to be considered as a characteristic energy scale, rather than as a temperature where the pseudogap is reduced to zero [34]. We believe that taking the $\rho(x)$ fluctuations into account may resolve the discrepancy between the experimental behavior of T^* and the temperature T_ρ introduced in the theory.

III. THE CORRELATION FUNCTION FOR THE PHASE FLUCTUATIONS

As stated above, we expect the phase fluctuations to be responsible for the difference between properties of the charged and neutral fermions defined above. The latter are described by the Green's function (2.15), which coincides with the BCS Green's function (2.5) only under the assumption that the phase θ of the order parameter $\Phi = \rho \exp(i\theta)$ is a constant and can be chosen to vanish. This is not the case for the 2D model, where there is a decay of the phase correlations and the Green's functions of charged and neutral fermions are nontrivially related via Eq. (2.12). To establish their relationship, one must know the correlator for the phase fluctuations. Its calculation is quite straightforward for $T < T_{\rm BKT}$, while for $T > T_{\rm BKT}$ one can apply the results of the BKT transition theory [35].

A. The correlator for $T < T_{\rm BKT}$

In the superconducting phase, the free vortex excitations are absent and the exponential correlator is easily expressed in terms of the Green's function

$$D_\theta(x) = \langle \theta(x) \theta(0) \rangle \tag{3.1}$$

(here, as above, $x \equiv \tau, \mathbf{r}$) via the Gaussian functional integral

$$\begin{aligned} D_{\alpha\beta}(x) &= \int \mathcal{D}\theta(x) \exp\left\{ -\int_0^{1/T} d\tau_1 \int d^2 r_1 \left[\frac{1}{2} \theta(x_1) D_\theta^{-1}(x_1) \theta(x_1) + I(x_1)\theta(x_1) \right] \right\} \\ &= \exp\left[-\frac{1}{2} \int_0^{1/T} d\tau_1 \int_0^{1/T} d\tau_2 \int d^2 r_1 \int d^2 r_2 I(\tau_1, \mathbf{r_1}) D_\theta(\tau_1 - \tau_2, \mathbf{r}_1 - \mathbf{r}_2) I(\tau_2, \mathbf{r_2}) \right], \end{aligned} \tag{3.2}$$

with the source

$$I(x_1) = -i\frac{\alpha}{2}\delta(\tau_1 - \tau)\delta(\mathbf{r}_1 - \mathbf{r}) + i\frac{\beta}{2}\delta(\tau_1)\delta(\mathbf{r}_1), \quad (\alpha, \beta = \pm). \tag{3.3}$$

The Green's function

$$D_\theta^{-1}(x) = -J(\mu, T, \rho)\nabla_r^2 - K(\mu, T, \rho)(\partial_\tau)^2 \tag{3.4}$$

for this model was found in [22]. Note that the superfluid stiffness J and compressibility K are here the functions of μ, T and ρ, and also that the Green's function (3.4) includes only the lowest derivatives of the phase θ. The higher terms are also present in the expansion, but we neglect them. In the simplest case where $J(\mu, T, \rho) \sim n_f$, the density of carriers, and $K(\mu, T, \rho) \sim \text{const}$ [22].

Substituting (3.4) into (3.2), we obtain

$$D_{\alpha\beta}(x) = \exp\left[-\frac{T}{4}\sum_{n=-\infty}^{\infty}\int\frac{d^2q}{(2\pi)^2}\frac{1 - \alpha\beta\cos(\mathbf{q}\mathbf{r} - \Omega_n\tau)}{Jq^2 + K\Omega_n^2}\right]. \tag{3.5}$$

It is easy to see that for zero frequency $\Omega_n = 0$, the integral in Eq. (3.5) is divergent at $\mathbf{q} = 0$ unless $\alpha = \beta$, and therefore only two terms survive in the sum over α, β in Eq. (2.12), namely

$$P_-\mathcal{G}(i\omega_n, \mathbf{k})P_- + P_+\mathcal{G}(i\omega_n, \mathbf{k})P_+ = -\frac{i\omega_n\hat{I} + \tau_3\xi(\mathbf{k})}{\omega_n^2 + \xi^2(\mathbf{k}) + \rho^2}. \tag{3.6}$$

It is important that the terms like $P_\pm G(i\omega_n, \mathbf{k})P_\mp$, which are proportional to τ_1 and thus violate the gauge symmetry, do not contribute to Eq. (2.12) due to vanishing of the corresponding D_{+-} and D_{-+} correlators standing after them. This explicitly demonstrates that the non-diagonal part of the 2D Green's function is vanishes at all finite temperatures. Thus, making use of the Gor'kov equations for the calculation of its diagonal part and the gap function is questionable. For nonzero correlators, we have

$$D(x) \equiv D_{++}(x) = D_{--}(x) = \exp\left[-\frac{T}{4}\sum_{n=-\infty}^{\infty}\int\frac{qdqd\varphi}{(2\pi)^2}\frac{1 - \cos(qr\cos\varphi)\cos\Omega_n\tau}{Jq^2 + K\Omega_n^2}\right]. \tag{3.7}$$

In what follows, we consider in detail only the static case $\tau = 0$. The restriction to this case is one of the few main assumptions we use throughout the paper.

The summation over n and the integration over φ in (3.7) can be readily done yielding the following exponent of (3.7)

$$-\frac{1}{16\pi\sqrt{JK}}\int_0^\infty dq e^{-q/\Lambda}[1 - J_0(qr)]\tanh\frac{qr_0}{4}, \tag{3.8}$$

where we introduced the scale

$$r_0 = \frac{2}{T}\sqrt{\frac{J}{K}}, \tag{3.9}$$

which is a function of the variables used (in the simplest case, $r_0 \sim \sqrt{n_f/T}$). In (3.8), we introduced the cutoff Λ by means of the exponential function. This cutoff represents the maximal possible momentum in the theory, the Brillouin momentum.

One can derive from (3.8) (see Appendix A) the following asymptotic expressions

$$D(0, \mathbf{r}) \sim \begin{cases} \left(\frac{r}{r_0}\right)^{-\dfrac{T}{8\pi J}}, & r \gg r_0 \gg \Lambda^{-1} \\ \left(\frac{\Lambda r}{2}\right)^{-\dfrac{T}{8\pi J}}, & r \gg \Lambda^{-1} \gg r_0. \end{cases} \tag{3.10}$$

This long-distance behavior governs the physics of θ-fluctuations that we intend to study in what follows.

We now discuss the meaning of the value r_0. Using again the phase stiffness $J(T = 0)$ and compressibility K from [22] we readily obtain that $r_0 = 2\sqrt{\epsilon_F/m}/T$, which is the single-particle thermal de Broglie wavelength ($\epsilon_F = \pi n_f/m$ is the Fermi energy). Then, assuming that $T \sim T_{\mathrm{BKT}}$ and taking $T_{\mathrm{BKT}} \simeq \epsilon_F/8$ [21,22], we can estimate

$$r_0 \sim \frac{16}{\sqrt{\epsilon_F m}} = \frac{16\sqrt{2}}{k_F}, \tag{3.11}$$

where k_F is the Fermi momentum. The value of k_F for cuprates is less than the Brillouin momentum Λ, which is why the first case in (3.10) seems to be more relevant.

There is another way to estimate r_0: we can use the value $2\Delta/T_c$, and hence,

$$r_0 \sim \sqrt{2}\pi \frac{2\Delta}{T_c} \xi_0, \tag{3.12}$$

where $\xi_0 = v_F/(\pi\Delta)$ is the BCS coherence length. This shows that r_0 has the meaning of a coherence length, which appears to be rather natural since the minimal size the phase coherence region should be of the order of ξ_0. Since the coherence length in cuprates is larger than the lattice spacing Λ^{-1}, we again obtain that the first case in (3.10) applies. Therefore, for $T < T_{\mathrm{BKT}}$ and for static fluctuations, we have that

$$D(\mathbf{r}) = \left(\frac{r}{r_0}\right)^{-\dfrac{T}{8\pi J}}, \tag{3.13}$$

where $r_0 = 16/\sqrt{\epsilon_F m}$.

B. The correlator for $T > T_{\mathrm{BKT}}$

For $T > T_{\mathrm{BKT}}$, the expression for static correlator (3.13) can be generalized using the well-known results of the BKT transition theory [35,36],

$$D(\mathbf{r}) = \left(\frac{r}{r_0}\right)^{-\dfrac{T}{8\pi J}} \exp\left(-\frac{r}{\xi_+(T)}\right), \tag{3.14}$$

where

$$\xi_+(T) = C \exp \sqrt{\frac{T_\rho - T}{T - T_{\mathrm{BKT}}}} \tag{3.15}$$

is the BKT coherence length and C is a constant whose value is discussed later. One can consider Eq.(3.14) as a general representation for $D(\mathbf{r})$ for both $T > T_{\mathrm{BKT}}$ and $T < T_{\mathrm{BKT}}$ if the coherence length $\xi_+(T)$ is considered to be infinite for $T < T_{\mathrm{BKT}}$. The pre–factor in Eq. (3.14) is related to the longitudinal (spin-wave) phase fluctuations, while the exponent is responsible for the transverse (vortex) excitations, which are present only above T_{BKT}. The pre–factor appears to be important for a non-Fermi liquid behavior discussed in what follows. Note, however, that the longitudinal phase fluctuations can be suppressed by the Coulomb interaction [30] that is not included in the present simple model. One further comment is that while the approximation used to study the vortex fluctuations in [30] is good for T well above T_{BKT}, the form of the correlator D is appropriate for T close to T_{BKT}.

The constant C can be estimated from the condition that $\xi_+(T)$ cannot be much less than the parameter r_0 which is a natural cutoff in the theory and we thus take $C = r_0/4$ in our numerical calculations $C = r_0/4$. In any case, for $T \gtrsim T_{\mathrm{BKT}}$, where (3.15) is valid, the value $\xi_+(T)$ is large and not very sensitive to the initial value of C.

There also exists a dynamical generalization of (3.14) proposed from phenomenological backgrounds in [37],

$$D(t, \mathbf{r}) = \exp(-\gamma t) \left(\frac{r}{r_0}\right)^{-\dfrac{T}{8\pi J}} \exp\left(-\frac{r}{\xi_+(T)}\right). \tag{3.16}$$

Note that t is the real time and γ is the decay constant, and theherefore (3.16) is the retarded Green's function. We hope to consider the more general case of dynamical phase fluctuations (3.16) elsewehere.

C. The Fourier transform of $D(\mathbf{r})$

For the Fourier transform (2.14) of (3.14), we have

$$\begin{aligned} D(i\Omega_n, \mathbf{q}) &= \int_0^{1/T} d\tau \int d^2 r \exp\left(i\Omega_n \tau - i\mathbf{q}\mathbf{r}\right) (r/r_0)^{-T/8\pi J} \exp(-r/\xi_+(T)) \\ &= 2\pi \frac{\delta_{n,0}}{T} r_0^{T/8\pi J} \int_0^\infty dr r^{1-T/8\pi J} J_0(qr) \exp(-r/\xi_+(T)). \end{aligned} \tag{3.17}$$

The integral in (3.17) can be calculated (see, for example, [38]) with the result

$$D(i\Omega_n, \mathbf{q}) = \frac{\delta_{n,0}}{T} \frac{2\pi r_0^{2(1-\alpha)} \Gamma(2\alpha)}{[q^2 + (1/\xi_+)^2]^\alpha} {}_2F_1\left(\alpha, -\alpha + \frac{1}{2}; 1; \frac{q^2}{q^2 + (1/\xi_+)^2}\right). \tag{3.18}$$

The hypergeometric function $F(a, b; c; z)$ can be well approximated by a constant since it is slowly varying at all values of $\mathbf{q}$. As this constant, we can take the value of the hypergeometric function at $q = \infty$. Thus,

$$D(i\Omega_n, \mathbf{q}) = \frac{\delta_{n,0}}{T} A[q^2 + (1/\xi_+)^2]^{-\alpha}, \tag{3.19}$$

where

$$A \equiv \frac{4\pi\Gamma(\alpha)}{\Gamma(1-\alpha)}\left(\frac{2}{r_0}\right)^{2(\alpha-1)}, \qquad \alpha \equiv 1 - \frac{T}{16\pi J}. \tag{3.20}$$

It should be stressed that for $T > T_{\rm BKT}$, the parameter α quickly deviates from unity as ϵ_F decreases; in other words, the underdoped region has to reveal highly non-standard properties in comparison with the overdoped one.

Note that for $\xi_+^{-1} = 0$ ($T < T_{\rm BKT}$), Eq. (3.19) is an exact Fourier transform of the correlator (3.13).

One should take into account that even for $T < T_{\rm BKT}$, the propagator (3.19) does not have the canonical behavior $\sim 1/q^2$, which is typical, for example, for the Bogolyubov mode in dimensions $d > 2$. In 2D, the modes with a propagator $\sim 1/q^2$ would lead to severe infrared singularities [25]; to avoid them, these modes transform into softer ones ($\sim 1/q^{2\alpha}$, $\alpha < 1$).

Finally, substituting (3.6) and (3.19) in (2.12), we obtain

$$G(i\omega_n, \mathbf{k}) = -A\int \frac{d^2q}{(2\pi)^2}\frac{i\omega_n + \tau_3\xi(\mathbf{q})}{\omega_n^2 + \xi^2(\mathbf{q}) + \rho^2}\frac{1}{[(\mathbf{k}-\mathbf{q})^2 + (1/\xi_+)^2]^\alpha}. \tag{3.21}$$

The coincidence of the Matsubara frequency in the left- and right-hand sides of Eq. (3.21) is evidently related to the static approximation used in this paper. As we see in the next sections, the Green's function (3.21), spectral density, and the density of states can be evaluated exactly.

IV. THE DERIVATION OF THE FERMION GREEN'S FUNCTION

The fermion Green's function can be calculated analytically if we split the fermion part of (3.21) as

$$\frac{i\omega_n\hat{I} + \tau_3\xi(\mathbf{k})}{\omega_n^2 + \xi^2(\mathbf{k}) + \rho^{\mathbf{2}}} = \frac{A_1}{\xi(\mathbf{k}) + i\sqrt{\omega_n^2+\rho^2}} + \frac{A_2}{\xi(\mathbf{k}) - i\sqrt{\omega_n^2+\rho^2}}, \tag{4.1}$$

where

$$A_1 = \frac{1}{2}\left(\tau_3 - \frac{\omega_n}{\sqrt{\omega_n^2+\rho^2}}\right), \qquad A_2 = \frac{1}{2}\left(\tau_3 + \frac{\omega_n}{\sqrt{\omega_n^2+\rho^2}}\right). \tag{4.2}$$

Using the representations

$$\frac{1}{a \pm ib} = \mp i\int_0^\infty ds \exp\left[\pm is(a \pm ib)\right], \tag{4.3}$$

$$\frac{1}{c^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt t^{\alpha-1}e^{-ct} \tag{4.4}$$

and taking (4.1) into account, we can rewrite (3.21) as

$$G(i\omega_n,\mathbf{k}) = \frac{iA}{\Gamma(\alpha)}\int_0^\infty ds\int_0^\infty dt t^{\alpha-1}e^{-\xi_+^{-2}t-s\sqrt{\omega_n^2+\rho^2}}\times$$
$$\int\frac{d^2q}{(2\pi)^2}\left\{A_1\exp\left[is\frac{q^2}{2m}-i\mu s-(\mathbf{k}-\mathbf{q})^2t\right]-A_2\exp\left[-is\frac{q^2}{2m}+i\mu s-(\mathbf{k}-\mathbf{q})^2t\right]\right\}. \quad (4.5)$$

Note that the special form of the integral representation (4.3) (compare with representation (4.4)) guarantees that the Gaussian integral over q is well-defined independently of the sign of $\xi(\mathbf{q}) = \mathbf{q}^2/2m-\mu$. Now the Gaussian integration over momenta q in (4.5) can be done explicitly:

$$G(i\omega_n,\mathbf{k}) = \frac{iA}{4\pi\Gamma(\alpha)}\int_0^\infty ds\int_0^\infty dt t^{\alpha-1}e^{-\xi_+^{-2}t-s\sqrt{\omega_n^2+\rho^2}}\times$$
$$\left[\frac{A_1}{t-is/2m}\exp\left(i\frac{\mathbf{k}^2}{2m}\frac{st}{t-is/2m}-i\mu s\right)-\frac{A_2}{t+is/2m}\exp\left(-i\frac{\mathbf{k}^2}{2m}\frac{st}{t+is/2m}+i\mu s\right)\right]. \quad (4.6)$$

Changing the variables as $s\to 2ms$ and further as $t\to st$, we can integrate over s with the result

$$G(i\omega_n,\mathbf{k}) = \frac{imA}{2\pi}\int_0^\infty dt t^{\alpha-1}\left\{\frac{A_1(t-i)^{\alpha-1}}{\left[\xi_+^{-2}t(t-i)+2m\sqrt{\omega_n^2+\rho^2}(t-i)-it\mathbf{k}^2+2im\mu(t-i)\right]^\alpha}\right.$$
$$\left.-\frac{A_2(t+i)^{\alpha-1}}{\left[\xi_+^{-2}t(t+i)+2m\sqrt{\omega_n^2+\rho^2}(t+i)+it\mathbf{k}^2-2im\mu(t+i)\right]^\alpha}\right\}. \quad (4.7)$$

In the general case where $\xi_+^{-1}\neq 0$, the denominator of (4.7) is quadratic in t and some further transformations are needed. Replacing $t\to -iu$ and expanding the quadratic polynomial in the denominator, we have

$$G(i\omega_n,\mathbf{k}) = -\frac{Am\xi_+^{2\alpha}}{2\pi}\left\{\int_0^{i\infty}du\frac{A_1u^{\alpha-1}(u+1)^{\alpha-1}}{[(u+u_1)(u+u_2)]^\alpha}+\int_0^{-i\infty}du\frac{A_2u^{\alpha-1}(u+1)^{\alpha-1}}{[(u+\tilde{u}_1)(u+\tilde{u}_2)]^\alpha}\right\}, \quad (4.8)$$

where

$$u_1 = m\xi_+^2\left(\frac{k^2\xi_+^2+1}{2m\xi_+^2}-\mu+i\sqrt{\omega_n^2+\rho^2}+\sqrt{D}\right),$$
$$u_2 = m\xi_+^2\left(\frac{k^2\xi_+^2+1}{2m\xi_+^2}-\mu+i\sqrt{\omega_n^2+\rho^2}-\sqrt{D}\right) \quad (4.9)$$

with

$$D\equiv\left(\frac{k^2\xi_+^2+1}{2m\xi_+^2}-\mu+i\sqrt{\omega_n^2+\rho^2}\right)^2+\frac{2}{m\xi_+^2}(\mu-i\sqrt{\omega_n^2+\rho^2}) \quad (4.10)$$

and

$$\tilde{u}_i = u_i(\sqrt{\omega_n^2+\rho^2}\to-\sqrt{\omega_n^2+\rho^2}). \quad (4.11)$$

We can verify from (4.9) that $\text{Re}u_i > 0$ for $\mu < 0$, and therefore, we can rotate the integration contour to the real axis:

$$G(i\omega_n, \mathbf{k}) = -\frac{Am\xi_+^{2\alpha}}{2\pi}\left\{\int_0^\infty du \frac{A_1 u^{\alpha-1}(u+1)^{\alpha-1}}{[(u+u_1)(u+u_2)]^\alpha} + (\sqrt{\omega_n^2+\rho^2} \to -\sqrt{\omega_n^2+\rho^2})\right\}. \quad (4.12)$$

The integral representation (4.12) can then be analytically continued to $\mu > 0$. The change of the variable $z = u/(u+1)$ allows Eq. (4.12) to be expressed in terms of Appell's function [40]

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 \frac{z^{\alpha-1}(1-z)^{\gamma-\alpha-1}}{(1-zx)^\beta(1-zy)^{\beta'}}dz, \quad (4.13)$$

and hence,

$$G(i\omega_n, \mathbf{k}) = -\frac{Am\xi_+^{2\alpha}}{2\pi\alpha}\left[\frac{A_1}{(u_1u_2)^\alpha}F_1\left(\alpha, \alpha, \alpha; \alpha+1; \frac{u_1-1}{u_1}, \frac{u_2-1}{u_2}\right) + (\sqrt{\omega_n^2+\rho^2} \to -\sqrt{\omega_n^2+\rho^2})\right]. \quad (4.14)$$

For $T < T_{\text{BKT}}$, the BKT coherence length is infinite ($\xi_+^{-1} = 0$), which means $(u_1-1)/u_1 = 1$ in the first argument of the Appell's function. This allows us to apply the reduction formula [40]

$$F_1(\alpha, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')}\ {}_2F_1(\alpha, \beta; \gamma-\beta'; x) \quad (4.15)$$

and express the result via the hypergeometric function

$$G(i\omega_n, \mathbf{k}) = -\Gamma^2(\alpha)\left(\frac{2}{mr_0^2}\right)^{\alpha-1}\left\{\frac{A_1}{[-(\mu - i\sqrt{\omega_n^2+\rho^2})]^\alpha}{}_2F_1\left(\alpha, \alpha; 1; \frac{k^2/2m}{\mu - i\sqrt{\omega_n^2+\rho^2})}\right) + \frac{A_2}{[-(\mu + i\sqrt{\omega_n^2+\rho^2})]^\alpha}{}_2F_1\left(\alpha, \alpha; 1; \frac{k^2/2m}{\mu + i\sqrt{\omega_n^2+\rho^2})}\right)\right\}, \quad (4.16)$$

where we inserted the value of A from (3.20).

This completes our derivation of the temperature fermion Green's function.

A. The retarded fermion Green's function

To obtain the spectral density, we need to obtain the retarded real-time Green's function from the temperature Green function by means of analytical continuation $i\omega_n \to \omega + i0$, and where $\sqrt{\omega_n^2+\rho^2} \to i\sqrt{\omega^2-\rho^2}$. This results in the following rules (compare with (4.2), (4.9), (4.10))

$$A_1 \to \mathcal{A}_1 = \frac{1}{2}\left(\tau_3 + \frac{\omega}{\sqrt{\omega^2-\rho^2}}\right), \qquad A_2 \to \mathcal{A}_2 = \frac{1}{2}\left(\tau_3 - \frac{\omega}{\sqrt{\omega^2-\rho^2}}\right), \quad (4.17)$$

$$u_1 \to v_1 = m\xi_+^2\left(\frac{k^2\xi_+^2+1}{2m\xi_+^2} - \mu - \sqrt{\omega^2-\rho^2} + \sqrt{\mathcal{D}}\right),$$
$$u_2 \to v_2 = m\xi_+^2\left(\frac{k^2\xi_+^2+1}{2m\xi_+^2} - \mu - \sqrt{\omega^2-\rho^2} - \sqrt{\mathcal{D}}\right), \tag{4.18}$$

with

$$D \to \mathcal{D} = \left(\frac{k^2\xi_+^2+1}{2m\xi_+^2} - \mu - \sqrt{\omega^2-\rho^2}\right)^2 + \frac{2}{m\xi_+^2}(\mu + \sqrt{\omega^2-\rho^2}) \tag{4.19}$$

and

$$\tilde{v}_i = v_i(\sqrt{\omega^2-\rho^2} \to -\sqrt{\omega^2-\rho^2}). \tag{4.20}$$

For the retarded Green's function we thus have

$$G(\omega,\mathbf{k}) = -\frac{Am\xi_+^{2\alpha}}{2\pi\alpha}\left\{\frac{\mathcal{A}_1}{(v_1v_2)^\alpha}F_1\left(\alpha,\alpha,\alpha;\alpha+1;\frac{v_1-1}{v_1},\frac{v_2-1}{v_2}\right)\right.$$
$$\left. + (\sqrt{\omega^2-\rho^2} \to -\sqrt{\omega^2-\rho^2})\right\}. \tag{4.21}$$

It is easy to see that

$$v_1v_2 = -2m\xi_+^2(\mu + \sqrt{\omega^2-\rho^2}). \tag{4.22}$$

We now discuss the condition under which the imaginary part of $G(\omega+i0,\mathbf{k})$ is nonvanishing.

For $|\omega| < \rho$, we can see that $\tilde{v}_1 = v_1^*$, $\tilde{v}_2 = v_2^*$, and therefore $G(\omega,\mathbf{k})$ is real and $\mathrm{Im}G(\omega+i0,\mathbf{k}) = 0$. The case where $|\omega| > \rho$ is more complicated. It follows from the Appell's function transformation property [40]

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = (1-x)^{-\alpha}F_1\left(\alpha,\gamma-\beta-\beta',\beta',\gamma;\frac{x}{x-1},\frac{y-x}{1-x}\right). \tag{4.23}$$

that for real x and y, the function F_1 becomes complex if $x > 1$ or/and $y > 1$. This implies that $G(\omega,\mathbf{k})$ has an imaginary part if $v_1 < 0$ or/and $v_2 < 0$. Looking at the expressions (4.18) for v_1 and v_2, we can see that v_1 is always positive, while v_2 may be negative. This means that $G(\omega,\mathbf{k})$ has a nonvanishing imaginary part if $v_1v_2 < 0$. Using (4.22), the condition for the existence of a nonzero imaginary part of $G(\omega,\mathbf{k})$ can then be written as $\mu + \sqrt{\omega^2-\rho^2} > 0$.

B. The branch cut structure of $G(\omega,\mathbf{k})$ and a non-Fermi liquid behavior

We now consider the retarded fermion Green's function (4.16) for $T < T_{\mathrm{BKT}}$. Applying the analytic continuation rules from the previous subsection to Eq. (4.16), we obtain

$$G(\omega,\mathbf{k}) = -\Gamma^2(\alpha)\left(\frac{2}{mr_0^2}\right)^{\alpha-1}\left[\frac{\mathcal{A}_1}{[-(\mu+\sqrt{\omega^2-\rho^2})]^\alpha}{}_2F_1\left(\alpha,\alpha;1;\frac{k^2/2m}{\mu+\sqrt{\omega^2-\rho^2})}\right)\right.$$
$$\left. + \frac{\mathcal{A}_2}{[-(\mu-\sqrt{\omega^2-\rho^2})]^\alpha}{}_2F_1\left(\alpha,\alpha;1;\frac{k^2/2m}{\mu-\sqrt{\omega^2-\rho^2})}\right)\right]. \tag{4.24}$$

Near the quasiparticle peaks where $\omega \approx \pm E(\mathbf{k})$, the arguments of the hypergeometric function in (4.24) are close to 1. One can consider, for instance, the first hypergeometric function, then

$$z_1 \equiv \frac{k^2/2m}{\mu + \sqrt{\omega^2 - \rho^2}} \simeq 1. \tag{4.25}$$

Using the relation between the hypergeometric functions [40]

$$\begin{aligned} {}_2F_1(a, b; c; z) = & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b+1-c; 1-z) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c+1-a-b; 1-z)\,, \end{aligned} \tag{4.26}$$

we obtain that near $z_1 \simeq 1$,

$$G(\omega, \mathbf{k}) \sim -\Gamma^2(\alpha) \left(\frac{2}{m r_0^2} \right)^{\alpha - 1} \frac{\mathcal{A}_1}{[-(\mu + \sqrt{\omega^2 - \rho^2})]^{\alpha}} \left\{ \frac{\Gamma(1-2\alpha)}{\Gamma^2(1-\alpha)} + \frac{\Gamma(2\alpha - 1)}{\Gamma^2(\alpha)} \frac{1}{(1-z_1)^{2\alpha - 1}} \right\}. \tag{4.27}$$

It can be seen that the expression for the Green's function obtained is evidently a nonstandard one: besides containing a branch cut, it clearly displays its non-pole character. The latter in its turn corresponds to a non-Fermi liquid behavior of the system as a whole. It must be stressed that non-Fermi liquid peculiarities are tightly related to the charge (i.e., observable) fermions only, because the Green's function (2.15) of neutral fermions has a typical (pole type) BCS form. In addition, it follows from (4.27) that new properties appear as a consequence of the θ-particle presence (leading to $\alpha \neq 1$), and because the parameter α is a function of T (see (3.20)), the non-Fermi liquid behavior is developed with temperature increase and is preserved until ρ vanishes.

It is interesting that in Anderson's theory [41], it was postulated that the Fermi liquid theory is broken down in the normal state as a result of strong correlations. Here, we started from the Fermi liquid theory and found that it is broken down due to strong phase fluctuations. As suggested in [41], the non-Fermi liquid behavior may lead to the suppression of the coherent tunneling between layers, which in turn confines carriers in the layers and leads to the strong phase fluctuations. In contrast to [41], however, our model predicts the restoration of the Fermi liquid behavior as T decreases, since $\alpha \to 1$ as $T \to 0$ (see the discussion in Sec. V C item 4.).

The $T = 0$ limit can also be obtained as follows. Strictly speaking, one cannot estimate the value of r_0 in the limit as $T \to 0$ in (4.24) via Eq. (3.11), because the substitution of $T_{\mathrm{BKT}} \simeq \epsilon_F/8$ in (3.9) is not valid in this case. However, this is not essential because $T/8\pi J \to 0$, so that the correlator (3.13), $D(\mathbf{r}) \to 1$, which evidently means the formation of a long-range order in the system. Furthermore, the value of α in (3.20) goes to 1 as $T \to 0$, and the hypergeometric function in (4.24) reduces to the geometrical series,

$${}_2F_1(1, 1; 1; z) = \frac{1}{1-z}. \tag{4.28}$$

Therefore, inserting (4.28) in (4.24), we obtain the standard BCS expression

$$G_{11}(\omega,\mathbf{k}) = \frac{\omega+\xi(\mathbf{k})}{\omega^2-\xi^2(\mathbf{k})-\rho^2}\,, \tag{4.29}$$

for the diagonal component $G_{11}(\omega,\mathbf{k})$ of the Nambu-Gor'kov Green's function $G(\omega,\mathbf{k})$.

Evidently Eq.(4.29) results in the standard BCS spectral density [31] with two δ-function peaks

$$A(\omega,\mathbf{k}) = \frac{1}{2}\left[1+\frac{\xi(\mathbf{k})}{E(\mathbf{k})}\right]\delta(\omega-E(\mathbf{k}))+\frac{1}{2}\left[1-\frac{\xi(\mathbf{k})}{E(\mathbf{k})}\right]\delta(\omega+E(\mathbf{k})), \tag{4.30}$$

where $E(\mathbf{k})=\sqrt{\xi^2(\mathbf{k})+\rho^2}$. To recover the nondiagonal components of G, one has to restore the correlators $D_{-+}(\mathbf{r})$ and $D_{+-}(\mathbf{r})$ that were omitted in Sec. III A.

V. THE SPECTRAL DENSITY OF THE FERMION GREEN'S FUNCTION

As is well known, [31], the spectral features of any system are entirely controlled by its spectral density

$$A(\omega,\mathbf{k}) = -\frac{1}{\pi}\mathrm{Im}G_{11}(\omega+i0,\mathbf{k})\,, \tag{5.1}$$

which, for example, for cuprates is measured in ARPES experiments (see [29]). This function defines the spectrum anisotropy, the presence of a gap, the DOS, etc. In what follows, we calculate $A(\omega,\mathbf{k})$ for the Green's function obtained above.

A. Analytical expression for the spectral density

For $v_1>0$ and $v_2<0$, the retarded fermion Green's function (4.21) can be rewritten (see Appendix B) as

$$\begin{aligned}G(\omega,\mathbf{k}) = &-\frac{Am\xi_+^{2\alpha}}{2\pi}\Bigg\{\mathcal{A}_1\left[\frac{(-1)^\alpha\Gamma(\alpha)\Gamma(1-\alpha)}{[v_1(1-v_2)]^\alpha}{}_2F_1\left(\alpha,\alpha;1;\frac{v_2(1-v_1)}{v_1(1-v_2)}\right)+\right.\\ &\left.\frac{1}{|v_2|}\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}F_1\left(1,\alpha,1-\alpha;2-\alpha;\frac{v_1}{v_2},\frac{1}{u_2}\right)\right]+(\sqrt{\omega^2-\rho^2}\to-\sqrt{\omega^2-\rho^2})\Bigg\}.\end{aligned} \tag{5.2}$$

Then, according to (5.1) the spectral density for the Green's function (5.2) has the form

$$\begin{aligned}A(\omega,\mathbf{k}) = &\frac{Am\xi_+^{2\alpha}\sin(\pi\alpha)}{2\pi^2}\mathrm{sgn}\omega\,\theta(\omega^2-\rho^2)\left[(\mathcal{A}_1)_{11}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{[v_1(1-v_2)]^\alpha}\right.\\ &\left.\times\,{}_2F_1\left(\alpha,\alpha;1;\frac{v_2(1-v_1)}{v_1(1-v_2)}\right)\theta(\mu+\sqrt{\omega^2-\rho^2})-(\sqrt{\omega^2-\rho^2}\to-\sqrt{\omega^2-\rho^2})\right].\end{aligned} \tag{5.3}$$

Using the quadratic transformation for the hypergeometric function [40]

$${}_2F_1(a,b;a-b+1;z) = (1-z)^{-a}{}_2F_1\left(\frac{a}{2},-b+\frac{a+1}{2};1+a-b;-\frac{4z}{(1-z)^2}\right)\,, \tag{5.4}$$

the expression (3.20) for A, Eqs. (4.18), and (4.19) we finally obtain

$$
\begin{aligned}
A(\omega, \mathbf{k}) = & \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} \left(\frac{2}{m r_0^2}\right)^{\alpha-1} \operatorname{sgn}\omega\, \theta(\omega^2 - \rho^2) \times \\
& \left[\frac{(\mathcal{A}_1)_{11}}{\mathcal{D}^{\alpha/2}} {}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; -4\frac{\frac{k^2}{2m}(\mu + \sqrt{\omega^2-\rho^2})}{\mathcal{D}}\right) \theta(\mu + \sqrt{\omega^2 - \rho^2}) \right. \\
& \left. -(\sqrt{\omega^2-\rho^2} \to -\sqrt{\omega^2-\rho^2}) \right], \qquad (5.5)
\end{aligned}
$$

where the chemical potential μ can be, in principle, determined from the equation that fixes the carrier density [22]. Here, however, we assume that the carrier density is sufficiently high and $\mu = \epsilon_F$.

In the BCS theory, $A(\omega, \mathbf{k})$ given by Eq. (4.30) consists of two pieces that are the spectral weights of adding and removing a fermion from the system respectively. Note that our splitting of $A(\omega, \mathbf{k})$ is different since each term in (5.5) corresponds to both the addition and the removal of a fermion.

In the next subsections, we verify the sum rule for (5.5), plot it for different temperatures, and discuss the results.

B. The sum rule for the spectral density

It is well known that for the exact Green's function $G(\omega, \mathbf{k})$, the spectral function (5.1) must satisfy the sum rule

$$
\int_{-\infty}^{\infty} d\omega A(\omega, \mathbf{k}) = 1. \qquad (5.6)
$$

The Green's function calculated in (4.21) is, of course, approximate. This is related to the use of the long-distance asymptotic behavior (3.10) of the phase correlator (3.7). This means that its Fourier transform (3.19) is, strictly speaking, valid for small $\mathbf{k}$ only, while we have integrated our expressions to the infinity. Another approximation that we have made was the restriction to the static phase fluctuations. Thus, it is important to check whether the sum rule (5.6) is satisfied with sufficient accuracy.

It is remarkable that for (5.5), the sum rule (5.6) can be tested analytically with the help of the techniques used in calculating $N(\omega)$ in Appendix C. We obtain

$$
\int_{-\infty}^{\infty} d\omega A(\omega, \mathbf{k}) = \frac{\Gamma(\alpha)}{\Gamma(2-\alpha)}. \qquad (5.7)
$$

The numerical value of the integral at the temperatures of interest can be estimated as follows. Taking the phase stiffness $J = 2/\pi T_{\mathrm{BKT}}$ at $T = T_{\mathrm{BKT}}$, the value α from (3.20) is given by

$$
\alpha \simeq 1 - \frac{1}{32}\frac{T}{T_{\mathrm{BKT}}}, \qquad T \sim T_{\mathrm{BKT}} \qquad (5.8)
$$

for T close to T_{BKT}. In particular, $\alpha(T = T_{\mathrm{BKT}}) = 31/32$ gives the following estimate for the right-hand side of (5.7), $\Gamma(\alpha)/\Gamma(2-\alpha) \simeq 1.037$. This shows that for $T \sim T_{\mathrm{BKT}}$, the spectral density (5.5) is reasonably good at the temperatures of interest.

The parameter α can however differ strongly from unity at $T > T_{\mathrm{BKT}}$ and in the underdoped regime.

C. Results for the spectral density

The plots of the spectral density $A(\omega, \mathbf{k})$ given by (5.5) at temperatures below and above T_{BKT} are presented in Figs.1–3. To draw these plots, we used the value of α from Eq. (5.8) and the mean-field value of ρ obtained from the corresponding equation in [21,22]. From these figures and our analytical expressions, we can infer the following results:

1. For $T < T_{\mathrm{BKT}}$ (the case presented on Fig. 1), there are two highly pronounced quasiparticle peaks at $\omega = \pm E(\mathbf{k})$. They are simply related to the contribution of zeros of $\mathcal{D}$ (see Eq. (4.19)) to $A(\omega, \mathbf{k})$.

2. We also observe two peaks at $\omega = \pm\rho$ when $k \neq k_F$ (for $k = k_F$, the value $E(\mathbf{k}_F) = \rho$, so that the two sets of peaks coincide). One can check that the divergence at these points is weaker than at the former peaks at $\omega = \pm E(\mathbf{k})$. In fact, these peaks are the result of the static and nointeracting approximation for the phase fluctuations used here. They are essential to satisfy the sum rule (5.6).

   If the dynamical fluctuations are taken into account, it is clear that the "external" frequency ω in $A(\omega, \mathbf{k})$ is different from the "internal" frequency in $\mathcal{A}_1, \mathcal{A}_2$ (see the discussion after Eq. (3.21) and compare it with Eq. (2.12)). We believe that this additional summation over the"internal" frequency (which is present if the dynamical fluctuations are considered) would considerably smear these peaks, moving the excess of the spectral weight inside the gap. This assumption is supported by the results of [37] (see item 3 below). The same effect can also be reached when the interaction between the charge and spin degrees of the freedom is taken into account [30]. Note also that the full cancellation of these peaks takes place in the $T = 0$ case given by Eq. (4.29).

3. For $\omega < |\rho|$, we have $A(\omega, \mathbf{k}) = 0$ and a gap exists at all T (including $T > T_{\mathrm{BKT}}$). This result is also a consequence of the static approximation used above. The dynamical fluctuations should fill the empty region resulting in the pseudogap formation in the normal state. Indeed, a filling of the gap was obtained in a related calculation [37] where the correlator $\langle \exp(i\theta(\mathbf{r}, t)\exp(-i\theta(0))\rangle$ (which differs from (3.16) only by the factor 1/2 muliplying the phase), which includes the dynamical phase fluctuations, was used in the numerical calculation of the self-energy of fermions and in the subsequent extraction of the spectral function from the fermion Green's function.

   In the approximation used in the present paper, the spin and charge degrees of freedom are decoupled (see Eq. (2.8)). However, this coupling can be included at the next stage of approximation and also leads to a pseudogap filling. Indeed, using the special form of the scattering rate proposed in [39], it was obtained in [30] that $A(\omega, \mathbf{k}) \neq 0$ even

for $\omega < |\rho|$. On the other hand, as stated above, there also are indications [37] that a filling of the gap can be obtained by considering the dynamical phase fluctuations only. At present, it is not clear which of these gap filling mechanisms plays the main role; this is the subject of our current investigations.

4. The main peaks at $\omega = \pm E(\mathbf{k})$ have a finite temperature dependent width which is, of course, related to the spin-wave (longitudinal) phase fluctuations. As $T \to 0$, the width goes to zero, but this limit cannot be correctly derived from (5.5) because this is an ordinary function, while the BCS spectral density (4.30) is a distribution. The correct limit can, however, be obtained for the integral of $A(\omega, \mathbf{k})$ (see Sec. VI, where the density of states is discussed). This sharpening of the peaks with decreasing T in the superconducting state was experimentally observed [29] and represents a striking difference from the BCS "pile-up" (4.30) which are present for all $T < T_c$.

   It was pointed out in [30] that the broadening of the spectral function caused by these fluctuations can be greater than the experimental data permits. This leads [30] to the conclusion that the spin-wave phase fluctuations are probably suppressed by the Coulomb interaction.

5. For $T > T_{\mathrm{BKT}}$ (see Figs. 2,3), one can see that the quasiparticle peaks at $\omega \approx \pm E(\mathbf{k})$ are less pronounced as the temperature increases. Indeed, the value of $A(\omega, \mathbf{k})$ at $\omega = \pm E(\mathbf{k})$ is, in contrast to the case where $T < T_{\mathrm{BKT}}$, already finite. This is caused by the fact that $\mathcal{D} \neq 0$ since ξ_+ is already finite due to the influence of the vortex fluctuations. As the temperature is increases further, ξ_+ decreases, so that the quasiparticle peaks disappear (compare Figs. 2 and 3). This behavior qualitatively reproduces the ARPES studies of the cuprates for the anti-node direction [29] (see also [42]) which show that the quasiparticle spectral function broadens dramatically when passing from the superconducting to normal state.

6. It is important to stress that due to a very smooth dependence of ξ_+^{-1} on T (see Eq. (3.15)) as the temperature varies from $T < T_{\mathrm{BKT}}$ to $T > T_{\mathrm{BKT}}$, there is no sharp transition at the point $T = T_{\mathrm{BKT}}$. There is a smooth evolution of the superconducting (excitation) gap $\Delta_{\mathrm{SC}} = \rho$ into the gap Δ_{PG}, which also is equal to ρ and in fact can be called a pseudogap because the system is not superconducting at $T > T_{\mathrm{BKT}}$. This qualitatively fits the experiment [29,34,42] and appears to be completely different from the BCS theory [31], where the gap vanishes at $T = T_c$. As was already mentioned, the gap obtained at $T > T_{\mathrm{BKT}}$ takes place in the static approximation only and begins to be filled after dynamical fluctuations are taken into account (see, for example, [37]).

7. Again for $T > T_{\mathrm{BKT}}$, one has $A(\omega, \mathbf{k}) = 0$ when $|\omega| < \rho$, and we expect the gapped region to be filled by the dynamical phase fluctuations [37]. We predict, however, an essential difference between the filling of the gap at $T > T_{\mathrm{BKT}}$ and $T < T_{\mathrm{BKT}}$. Indeed, due to the presence of the vortices above T_{BKT}, the value of the decay constant γ in Eq. (3.16) should be much larger than for $T < T_{\mathrm{BKT}}$. This and a nonzero value of ξ_+^{-1} above T_{BKT} may explain the break at $T = T_c$ in the scattering rate Γ_1 introduced in [39]. In general, it is interesting to establish a correspondence between the phenomenological parameters, Γ_1 and Γ_0 introduced in [39] and the vortex parameters ξ_+ and γ used

here. Note, however, that this correspondence cannot be simple because of the non-pole character of the Green's function derived here.

As mentioned above (see item 3), the filling of the gap due to dynamical phase fluctuations is not the only possible mechanism for filling and the presence of vortices above T_{BKT} can be taken into account via coupling the spin and charge degrees of freedom [30]. It could also be that both these mechanisms are physically equivalent, since they relate the gap filling to the presence of vortices in the system.

8. Since we used the mean-field dependence $\rho(T)$, it is clear that the distance between the quasiparticle peaks (which is approximately equal to 2ρ) diminishes as T increases. This process of the pseudogap closing is accompanied by the destruction of the quasiparticle peaks. It is evident also that for $\rho = 0$, the normal Fermi liquid behavior is immediately restored because $J(\rho = 0) = 0$ [21,22]. Recall, however, that the description proposed here cannot be applied when ρ is rather small, because, as already mentioned, the fluctuations of $\rho(x)$ have to be also taken into account in this region.

VI. THE DENSITY OF STATES

The density of states can be found from the formula

$$N(\omega) = \int \frac{d^2k}{(2\pi)^2} A(\omega, \mathbf{k}) = N_0 \int_0^W d\frac{k^2}{2m} A(\omega, \mathbf{k}), \tag{6.1}$$

where $N_0 \equiv m/2\pi$ is the density of 2D states in the normal state (W is the bandwidth).

This integral can be calculated analytically (see Appendix C), and which gives

$$\begin{aligned} N(\omega) = N_0 \frac{\Gamma(\alpha)}{\Gamma(2-\alpha)} \left(\frac{2}{mr_0^2}\right)^{\alpha-1} \mathrm{sgn}\omega\, \theta(\omega^2 - \rho^2) \times \\ \left\{ (\mathcal{A}_1)_{11} \left[\left(\frac{1}{2m\xi_+^2} + W - \mu - \sqrt{\omega^2 - \rho^2} \right)^{1-\alpha} - \left(\frac{1}{2m\xi_+^2} \right)^{1-\alpha} \right] \theta(\mu + \sqrt{\omega^2 - \rho^2}) \right. \\ \left. - \left(\sqrt{\omega^2 - \rho^2} \to -\sqrt{\omega^2 - \rho^2} \right) \right\}. \end{aligned} \tag{6.2}$$

Again for $T = 0$ and large $\mu \gg \rho$, Eq. (6.2) reduces to the BCS result [31]

$$N(\omega) = N_0 \frac{|\omega|}{\sqrt{\omega^2 - \rho^2}}. \tag{6.3}$$

The plots for DOS (6.2) are presented in Fig. 4 ($T < T_{\mathrm{BKT}}$) and Figs. 5,6 for $T > T_{\mathrm{BKT}}$, respectively. If one does not pay attention to a small difference in the curves shown in these figures, it is well seen that qualitatively the form of DOS does not differ from standard BCS curves. Moreover, similarly to the spectral function, the DOS in the static approximation has a gap both above and below T_{BKT} and does not reveal any change when the temperature crosses the phase transition point. This confirms once more the crossover character of the latter, although, as was already pointed out, a 2D system is superconducting below T_{BKT} only. According to generally accepted views, the existence of an empty gap above the

critical temperature is impossible. The reasons for its persistence were discussed in the previous section. Recall only that dynamical fluctuations or fluctuations of the modulus ρ undoubtedly result in the gap filling above $T_{\rm BKT}$. One must also take the dependence of the decay constant γ into account (see (3.16)), which for $T > T_{\rm BKT}$ can be essentially bigger than in the region $T < T_{\rm BKT}$ due to the presence of the vortices.

From the physical point of view, the filling of a gap (transforming it into a pseudogap) above $T_{\rm BKT}$ (or T_c in quasi-2D case) has to continue up to T^* (or T_ρ if there is a point where $\rho = 0$). However, taking ρ-fluctuations into account (i.e., $\rho(x) \to \rho + \Delta\rho$) will cause the appearance of the self-energy, in addition to ρ^2, in the denominator of the mean field Green's function (2.15); it is proportional to the quantity $\langle \Delta\rho(x)\Delta\rho(0)\rangle$ whose contribution could persist at all T. In this case, the beginning of the pseudogap opening will be defined by the experimental technique sensitivity of the spectral function or DOS measurement.

VII. CONCLUSION

To summarize, we have derived analytic expressions for the fermion Green's function, its spectral density, and the density of states in the modulus-phase representation for the simplest 2D attractive Hubbard model with the s-wave nonretarded attractive interaction.

While there is still no generally accepted microscopic theory of HTSC compounds and their basic features (including the pairing mechanism), it seems that this approach, although in a sense phenomenological, is of great interest since it enables one to propose a reasonable interpretation for the pseudogap phenomena related to the vortex fluctuations. The results presented here are entirely analytic, which allows a deeper understanding than in the case of a numerical investigation. In particular, the analytic investigation of the Green's function structure revealed that the phase fluctuations lead to a non-Fermi liquid behavior below and above $T_{\rm BKT}$.

Evidently, there are a number of important open questions. The main question is whether the pseudogap is related to some kind of superconducting (in our case, phase) fluctuations. Hopefully, the experiment proposed in [4] may answer this question. It seems plausible from the theoretical point of view that superconducting fluctuations should contribute to the pseudogap (see, however, [11]). Nevertheless, one cannot exclude the possibility that the superconducting contribution may be neither the only nor the main contribution.

Another open question is which approach allows one to obtain the pseudogap from the attractive Hubbard model. The schemes used in [30] and in our paper are very different from those of [13]. In particular, our approach allowed us to establish a direct relationship between the superconducting fluctuations and the non-Fermi liquid behavior in a very natural and transparent way. Also, it relates the pseudogap to the "soup" of fluctuating vortices (see also [30,43]), while [13] emphasises the existence of metastable pairs above T_c. It is possible that both these pictures capture some physics, but in the different regions of the temperatures. When T is high and close to T^*, the value of ρ is small, so that ρ-fluctuations or metastable pairs dominate. Then as the temperature approaches $T_{\rm BKT}$, the values of ρ and the phase stiffness J are growing bigger, so that the vortex excitations dominate and ρ-fluctuations become less important. We stress once more that the vortex excitations cannot be adequately described within T-matrix approximation [9].

Recently the last part of this picture was supported experimentally [44] by the measurements of the screening and dissipation of a high-frequency electromagnetic field in bismuth-cuprate films. These measurements provide evidence for a phase-fluctuation driven transition from the superconducting to normal state.

Finally, there remains the problem of a more complete treatment of the pseudogap in the modulus-phase variables. In particular, the effects of dynamical phase fluctuations and the fluctuations of the order field modulus must be considered. The latter are especially important for the d-wave superconductor since the modulus can be arbitrary small in the nodal directions. In this case it will be important again to check the complete structure of the Green's function, especially its non-pole structure. Another important question that has to be addressed is which factor is more important for the gap filling, the spin-charge coupling proposed in [30] or dynamical phase fluctuations which, as was shown in [37], also result in filling.

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APPENDIX A: THE ASYMPTOTIC OF THE PHASE CORRELATOR

To calculate the integral in Eq. (3.8), we first write it as

$$\begin{aligned} I \equiv & \int_0^\infty e^{-q/\Lambda}[1 - J_0(qr)] \coth aq = \\ & \frac{1}{a}\int_0^\infty dt e^{-t/\Lambda a}\left(\coth t - \frac{1}{t}\right)\left[1 - J_0\left(\frac{r}{a}t\right)\right] + \frac{1}{a}\int_0^\infty \frac{dt}{t} e^{-t/\Lambda}[1 - J_0(rt)]. \end{aligned} \tag{A.1}$$

The following formulas are used when calculating I:

$$\begin{aligned} & \int_0^\infty dt e^{-\beta t}\left(\coth t - \frac{1}{t}\right) = \ln\frac{\beta}{2} + \frac{1}{\beta} - \psi\left(1 + \frac{\beta}{2}\right); \\ & \int_0^\infty \frac{dt}{t} e^{-pt}[1 - J_0(ct)] = \ln\frac{p + \sqrt{p^2 + c^2}}{2p}. \end{aligned} \tag{A.2}$$

Hence, we obtain

$$\begin{aligned} I = & \frac{4}{r_0}\left[\ln\frac{1 + \sqrt{1 + (\Lambda r)^2}}{\Lambda r_0} + \frac{\Lambda r_0}{4} - \psi\left(1 + \frac{2}{\Lambda r_0}\right)\right] - \frac{1}{r}\int_0^\infty dt e^{-t/\Lambda r}\left(\coth\frac{r_0 t}{4r} - \frac{4r}{r_0 t}\right) J_0(t) \\ \sim & \frac{4}{r_0}\left[\ln\frac{r}{r_0} + \frac{\Lambda r_0}{4} - \psi\left(1 + \frac{2}{\Lambda r_0}\right)\right] - \frac{1}{r}\frac{1}{\sqrt{1 + 1/(\Lambda r)^2}}, \qquad r \gg r_0, \Lambda^{-1}. \end{aligned} \tag{A.3}$$

Now, depending on the relationship between Λ and r_0, we obtain

$$I \sim \begin{cases} \dfrac{4}{r_0} \ln \dfrac{r}{R_0} + \Lambda, & r \gg r_0 \gg \Lambda^{-1} \\ \dfrac{4}{r_0} \ln \dfrac{\Lambda r}{2}, & r \gg \Lambda^{-1} \gg r_0, \end{cases} \tag{A.4}$$

which gives Eq. (3.10).

APPENDIX B: ANOTHER REPRESENTATION FOR THE RETARDED GREEN'S FUNCTION

Here, we obtain another representation for the retarded fermion Green's function that is more convenient for the derivation of the spectral density. Recall that when the imaginary part of $G(\omega, \mathbf{k})$ is nonzero, $\mu + \sqrt{\omega^2 - \rho^2} > 0$ and $v_1 > 0$, $v_2 < 0$. This allows one to transform the analytically continued (by means of Eq. (4.18)) integral

$$L \equiv \int_0^\infty du \frac{[u(u+1)]^{\alpha-1}}{[(u+v_1)(u+v_2)]^\alpha} \tag{B.1}$$

from Eq. (4.14) as (for $\alpha < 1$)

$$L = (-1)^\alpha \int_0^{|v_2|} du \frac{[u(u+1)]^{\alpha-1}}{[(u+v_1)(|v_2|-u)]^\alpha} + \int_{|v_2|}^\infty du \frac{[u(u+1)]^{\alpha-1}}{[(u+v_1)(u-|v_2|)]^\alpha} = \frac{(-1)^\alpha}{u_1^\alpha}\Gamma(\alpha) \times \\ \Gamma(1-\alpha) F_1\left(\alpha, \alpha, 1-\alpha; 1; \frac{v_2}{v_1}, u_2\right) + \frac{1}{|v_2|}\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} F_1\left(1, \alpha, 1-\alpha; 2-\alpha; \frac{v_1}{v_2}, \frac{1}{v_2}\right). \tag{B.2}$$

The first Appell function in (B.2) can be reduced to the hypergeometric function using the identity [40] taht is valid for $\gamma = \beta + \beta'$

$$F_1(\alpha, \beta, \beta', \beta+\beta'; x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta; \beta+\beta'; \frac{x-y}{1-y}\right). \tag{B.3}$$

Thus one obtains

$$L = \frac{(-1)^\alpha \Gamma(\alpha)\Gamma(1-\alpha)}{[u_1(1-u_2)]^\alpha} {}_2F_1\left(\alpha, \alpha; 1; \frac{u_2(1-u_1)}{u_1(1-u_2)}\right) + \frac{1}{|u_2|}\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \times \\ F_1\left(1, \alpha, 1-\alpha; 2-\alpha; \frac{u_1}{u_2}, \frac{1}{u_2}\right); \quad \frac{u_2(1-u_1)}{u_1(1-u_2)} < 1, \quad \frac{u_1}{u_2} < 0, \quad \frac{1}{u_2} < 0. \tag{B.4}$$

This completes the derivation of Eq. (5.2).

APPENDIX C: THE CALCULATION OF THE DENSITY OF STATES

Introducing

$$y = \frac{k^2/2m}{\mu + \sqrt{\omega^2 - \rho^2}}, \qquad b = \frac{1}{2m\xi_+^2}\frac{1}{\mu + \sqrt{\omega^2 - \rho^2}} \qquad y_0 = \frac{W}{\mu + \sqrt{\omega^2 - \rho^2}}, \tag{C.1}$$

and substituting (5.5) in (6.1), we can write

$$N(\omega) = N_0 \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} \left(\frac{2}{mr_0^2}\right)^{\alpha-1} \mathrm{sgn}\omega\, \theta(\omega^2-\rho^2) \left[(\mathcal{A}_1)_{11} (\mu + \sqrt{\omega^2-\rho^2})^{1-\alpha} \times \right.$$
$$\int_0^{y_0} \frac{dy}{[(y+b-1)^2+4b]^{\alpha/2}} {}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; -\frac{4y}{(y+b-1)^2+4b}\right)$$
$$\left. \times \theta(\mu + \sqrt{\omega^2-\rho^2}) - (\sqrt{\omega^2-\rho^2} \to -\sqrt{\omega^2-\rho^2}) \right]. \tag{C.2}$$

We now consider the integral from (C.2),

$$I = \int_0^{y_0} \frac{dy}{[(y+b-1)^2+4b]^{\alpha/2}} {}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; -\frac{4y}{(y+b-1)^2+4b}\right). \tag{C.3}$$

Using the relation [40]

$${}_2F_1(a,b;c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{C.4}$$

it can then be rewritten as

$$I = \int_0^{y_0} \frac{dy}{(y+b+1)^{\alpha}} {}_2F_1\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; 1; \frac{4y}{(y+b+1)^2}\right). \tag{C.5}$$

Replacing $x = \dfrac{b+1}{y+b+1}$ in (C.5), we obtain

$$I = (b+1)^{1-\alpha} \int_{x_0}^1 dx x^{\alpha-2} {}_2F_1\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; 1; \frac{4x(1-x)}{b+1}\right), \quad x_0 = \frac{b+1}{y_0+b+1}. \tag{C.6}$$

The integral (C.5) diverges at the lower limit as $x_0 \to 0$ or equivalently as $y_0 \to \infty$. To handle this we can write

$$I = (b+1)^{1-\alpha} \int_{x_0}^1 dx x^{\alpha-2} \left[{}_2F_1\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; 1; \frac{4x(1-x)}{b+1}\right) - 1 + 1 \right]$$
$$= (b+1)^{1-\alpha} \left\{ \frac{1-x_0^{\alpha-1}}{\alpha-1} + \int_{x_0}^1 dx x^{\alpha-2} \left[{}_2F_1\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; 1; \frac{4x(1-x)}{b+1}\right) - 1 \right] \right\}. \tag{C.7}$$

To calculate the last integral in (C.7), we rewrite it as

$$E = \lim_{\gamma \to \alpha-1} \int_0^1 dx x^{\gamma-1} \left[{}_2F_1\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; 1; \frac{4x(1-x)}{b+1}\right) - 1 \right]. \tag{C.8}$$

For $\gamma > 0$ we can compute the integral with the help of the formula (2.21.29) [45]

$$\int_0^y x^{\alpha-1}(y-x)^{\beta-1} {}_2F_1(a,b;c;\omega x(y-x)) dx$$
$$= y^{\alpha+\beta-1} B(\alpha,\beta) {}_4F_3\left(a,b,\alpha,\beta; c, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \frac{\omega y^4}{4}\right), \qquad y,\ \mathrm{Re}\alpha,\ \mathrm{Re}\beta > 0, \tag{C.9}$$

so that

$$
\begin{aligned}
E &= \lim_{\gamma\to\alpha-1}\left\{B(\gamma,1)_4F_3\left(\frac{\alpha}{2},\frac{1+\alpha}{2},\gamma,1;1,\frac{\gamma+1}{2},\frac{\gamma+2}{2};\frac{1}{b+1}\right)-\frac{1}{\gamma}\right\}\\
&= B(\alpha-1,1)_4F_3\left(\frac{\alpha}{2},\frac{1+\alpha}{2},\alpha-1,1;1,\frac{\alpha}{2},\frac{1+\alpha}{2};\frac{1}{b+1}\right)-\frac{1}{\alpha-1}\\
&= \frac{1}{\alpha-1}{}_1F_0\left(\alpha-1;\frac{1}{b+1}\right)-\frac{1}{\alpha-1}=\frac{1}{1-\alpha}\left[1-\left(\frac{b}{b+1}\right)^{1-\alpha}\right]>0.
\end{aligned}
\tag{C.10}
$$

Thus, for the integral (C.3) we find

$$
I=\frac{1}{1-\alpha}[(y_0+b+1)^{1-\alpha}-b^{1-\alpha}]. \tag{C.11}
$$

Now substituting (C.11) into (C.2) we obtain

$$
\begin{aligned}
N(\omega) &= N_0\frac{\Gamma(\alpha)}{\Gamma(2-\alpha)}\left(\frac{2}{mr_0^2}\right)^{\alpha-1}\mathrm{sgn}\omega\,\theta(\omega^2-\rho^2)\Big\{(\mathcal{A}_1)_{11}(\mu+\sqrt{\omega^2-\rho^2})^{1-\alpha}\\
&\times\Big[(y_0+b+1)^{1-\alpha}-b^{1-\alpha}\Big]\theta(\mu+\sqrt{\omega^2-\rho^2})-(\sqrt{\omega^2-\rho^2}\to-\sqrt{\omega^2-\rho^2})\Big\}.
\end{aligned}
\tag{C.12}
$$

Finally, replacing y_0 and b in (C.12) by expressions from (C.1) we arrive at Eq. (6.2).

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FIGURES

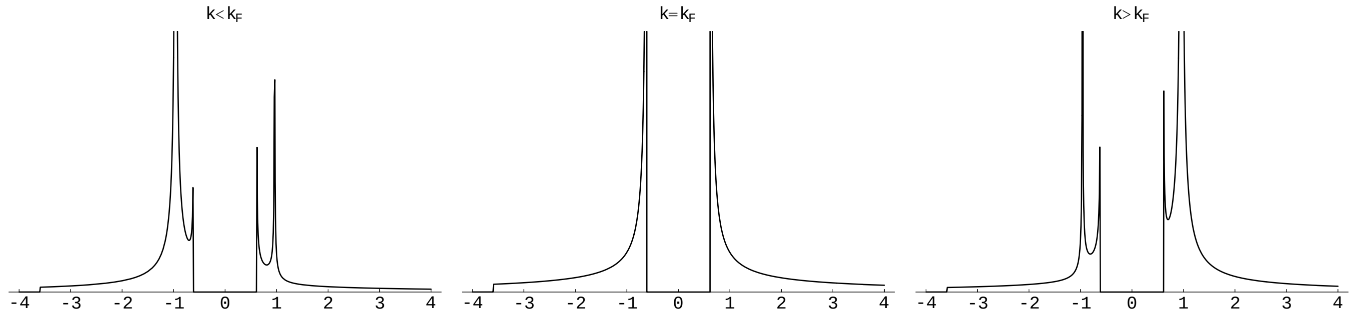

FIG. 1. Plot of the spectral function $A(\omega, \mathbf{k})$ as a function of ω in units of the zero temperature gap Δ for $k < k_F$, $k = k_F$ and $k > k_F$ at $T = 0.99 T_{\rm BKT}$.

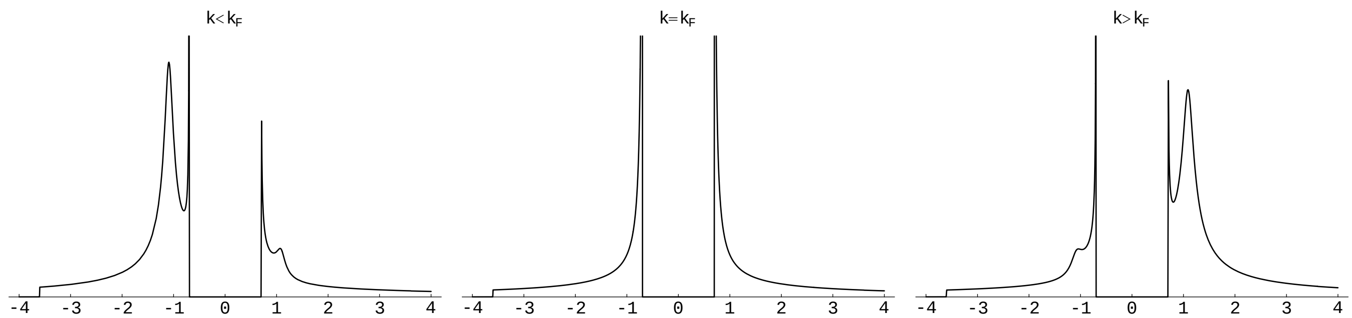

FIG. 2. Plot of the spectral function $A(\omega, \mathbf{k})$ as a function of ω in units of the zero temperature gap Δ for $k < k_F$, $k = k_F$ and $k > k_F$ at $T = 1.043 T_{\rm BKT}$.

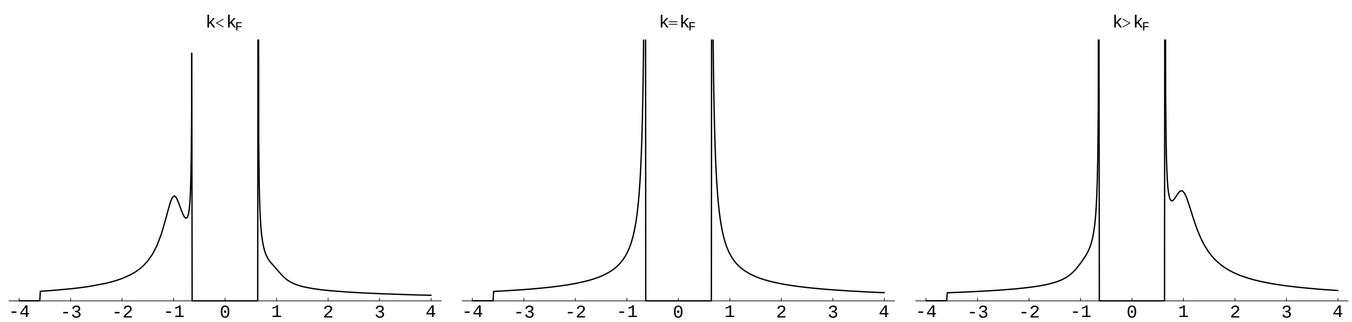

FIG. 3. Plot of the spectral function $A(\omega, \mathbf{k})$ as a function of ω in units of the zero temperature gap Δ for $k < k_F$, $k = k_F$ and $k > k_F$ at $T = 1.088 T_{\rm BKT}$.

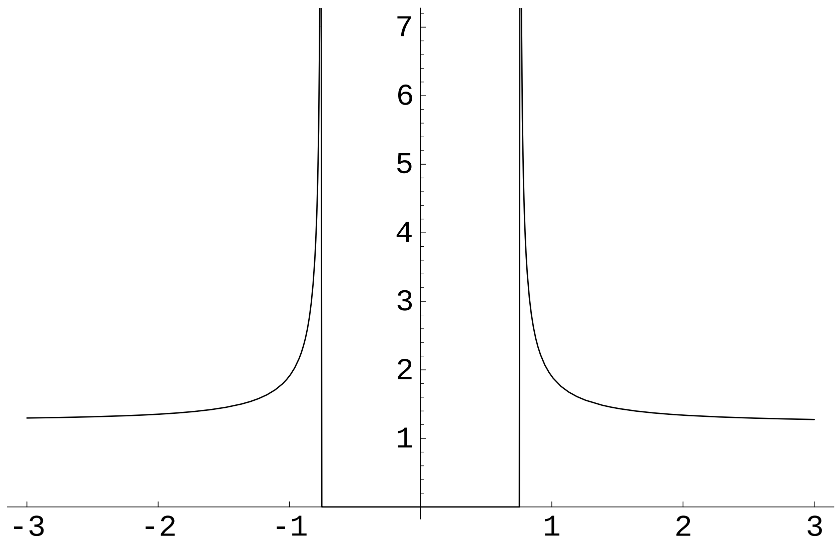

FIG. 4. The density of states $N(\omega)/N_0$ at $T = 0.99T_{\mathrm{BKT}}$.

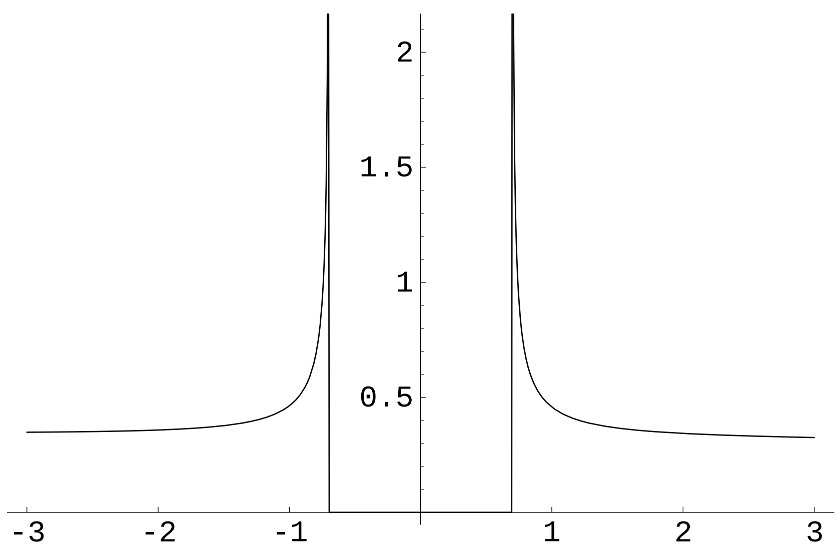

FIG. 5. The density of states $N(\omega)/N_0$ at $T = 1.043T_{\mathrm{BKT}}$.

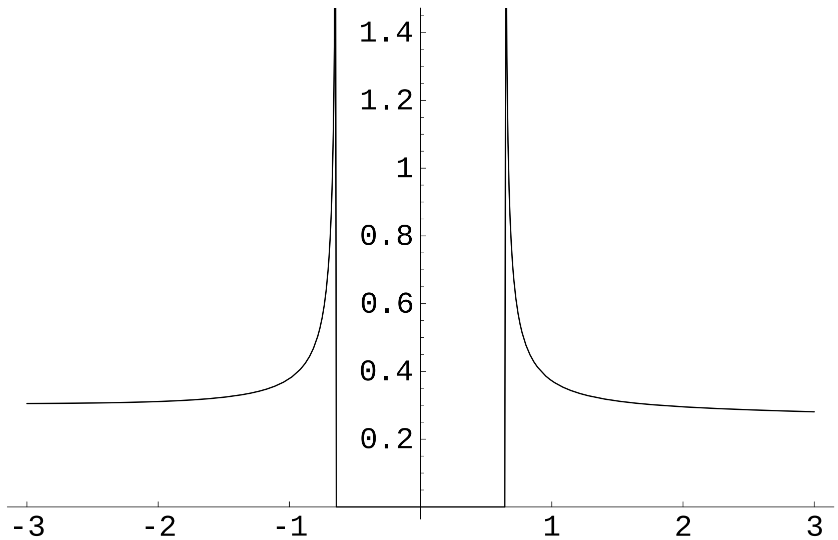

FIG. 6. The density of states $N(\omega)/N_0$ at $T = 1.088T_{\rm BKT}$.